CS103 Handout 36
Fall 2017 November 10, 2017

Problem Set 7

What can you do with regular expressions? What are the limits of regular languages? In this problem set, you'll explore the answers to these questions along with their practical consequences.

As always, please feel free to drop by office hours, ask on Piazza, or send us emails if you have any questions. We'd be happy to help out.

Good luck, and have fun!

Due Friday, November 17th at 2:30PM

Problem One: Designing Regular Expressions

Below are a list of alphabets and languages over those alphabets. For each language, write a regular expression for that language.

Please use our online tool to design, test, and submit your regular expressions. Typed or handwritten solutions will not be accepted. To use it, visit the CS103 website and click the "Regex Editor" link under the "Resources" header. As before, if you submit in a pair, please make a note in your GradeScope submission of which partner submitted your answers to this question so that we know where to look. Also, as a reminder, please test your submissions thoroughly, since we'll be grading them with an autograder.

i. Let $\Sigma = \{\texttt{a}, \texttt{b}\}$ and let $L = \{ w \in \Sigma^* \mid w$ does not contain `ba` as a substring $\}$. Write a regular expression for L.

ii. Let $\Sigma = \{\texttt{a}, \texttt{b}\}$ and let $L = \{ w \in \Sigma^* \mid w$ does not contain `bb` as a substring $\}$. Write a regular expression for L.

iii. Suppose you are taking a walk with your dog on a leash of length two. Let $\Sigma = \{\texttt{y}, \texttt{d}\}$ and let $L = \{ w \in \Sigma^* \mid w$ represents a walk with your dog on a leash where you and your dog both end up at the same location $\}$. For example, the string `yyddddyy` is in L because you and your dog are never more than two steps apart and both of you end up four steps ahead of where you started; similarly, `ddydyy` $\in L$. However, `yyyyddd` $\notin L$, since halfway through your walk you are three steps ahead of your dog; `ddyd` $\notin L$, because your dog ends up two steps ahead of you; and the string `ddyddyyy` $\notin L$, because at one point in your walk your dog is three steps ahead of you. Write a regular expression for L.

iv. Let $\Sigma = \{\texttt{a}, \texttt{b}\}$ and let $L = \{ w \in \Sigma^* \mid w \neq \texttt{ab} \}$. Write a regular expression for L.

v. Let $\Sigma = \{\texttt{M}, \texttt{D}, \texttt{C}, \texttt{L}, \texttt{X}, \texttt{V}, \texttt{I}\}$ and let $L = \{ w \in \Sigma^* \mid w$ is number less than 2,000 represented in Roman numerals $\}$. For example, `CMXCIX` $\in L$, since it represents the number 999, as are the strings `L` (50), `VIII` (8), `DCLXVI` (666), `CXXXVII` (137), `CDXII` (412), and `MDCXVIII` (1,618). However, we have `VIIII` $\notin L$ (you'll never have four `I`'s in a row; use `IX` or `IV` instead), that `MMI` $\notin L$ (it's a Roman numeral, but it's for 2,001, which is too large), that `VX` $\notin L$ (this isn't a valid Roman numeral), and that `IM` $\notin L$ (the notation of using a smaller digit to subtract from a larger one only lets you use `I` to prefix `V` and `X`, or `X` to prefix `L` and `C`, or `C` to prefix `D` and `M`). The Romans didn't have a way of expressing the number 0, so to make your life easier we'll say that $\varepsilon \in L$ and that the empty string represents 0. (Oh, those silly Romans.) Write a regular expression for L.

   (As a note, we're using the "standard form" of Roman numerals. You can see a sample of numbers written out this way via this link.)

Problem Two: Finite and Cofinite Languages

A language L is called *finite* if L contains finitely many strings. More precisely, a language L is a finite language if $|L|$ is a natural number. A language L is called *cofinite* if its complement is a finite language; that is, L is cofinite if $|\overline{L}|$ is a natural number.

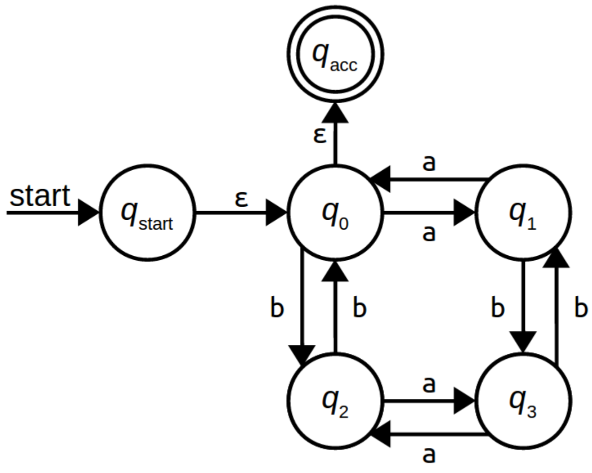
i. Prove that any finite language is regular.

ii. Prove that any cofinite language is regular.

Problem Three: State Elimination

The state elimination algorithm gives a way to transform a finite automaton (DFA or NFA) into a regular expression. It's a really beautiful algorithm once you get the hang of it, so we thought that we'd let you try it out on a particular example.

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{\ w \in \Sigma^* \mid w$ has an even number of a's and an even number of b's$\}$. Below is a finite automaton for L that we've prepared for the state elimination algorithm by adding in a new start state q_{start} and a new accept state q_{acc}:



We'd like you to use the state elimination algorithm to produce a regular expression for L.

i. Run two steps of the state elimination algorithm on the above automaton. Specifically, first remove state q_1, then remove state q_2. Show your result at this point.

ii. Finish the state elimination algorithm. What regular expression do you get for L?

iii. Without making reference to the original automaton given above, give an intuitive explanation for how the regular expression you found in part (ii) works.

Problem Four: Distinguishable Strings

The Myhill-Nerode theorem is one of the trickier and more nuanced theorems we've covered this quarter. This question explores what the theorem means and, importantly, what it *doesn't* mean.

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{\ w \in \Sigma^* \mid |w| \text{ is even }\}$.

i. Show that L is a regular language.

ii. Prove that there is a infinite set $S \subseteq \Sigma^*$ where there are infinitely many pairs of distinct strings $x, y \in S$ such that $x \not\equiv_L y$.

iii. Prove that there is *no* infinite set $S \subseteq \Sigma^*$ where *all* pairs of distinct strings $x, y \in S$ satisfy $x \not\equiv_L y$.

The distinction between parts (ii) and (iii) is important for understanding the Myhill-Nerode theorem. A language is nonregular not if you can find infinitely many pairs of distinguishable strings, but rather if you can find infinitely many strings that are all *pairwise* distinguishable. This is a subtle distinction, but it's an important one!

Problem Five: Balanced Parentheses

Consider the following language over $\Sigma = \{$`(`, `)`$\}$:

$$L_1 = \{ \ w \in \Sigma^* \mid w \text{ is a string of balanced parentheses } \}$$

For example, we have `()` $\in L_1$, `(())` $\in L_1$, `(()())()` $\in L_1$, $\varepsilon \in L_1$, and `(())((()()))` $\in L_1$, but `)(` $\notin L_1$, `(()` $\notin L_1$, and `((())))` $\notin L_1$. This question explores properties of this language.

i. Prove that L_1 is not a regular language. One consequence of this result – which you don't need to prove – is that most languages that support some sort of nested parentheses, such as most programming languages and HTML, aren't regular and so can't be parsed using regular expressions.

Let's say that the *nesting depth* of a string of balanced parentheses is the maximum number of unmatched open parentheses at any point inside the string. For example, the string `((()))` has nesting depth three, the string `(()())()` has nesting depth two, and the string ε has nesting depth zero.

Consider the language $L_2 = \{ \ w \in \Sigma^* \mid w$ is a string of balanced parentheses and w's nesting depth is at most four $\}$. For example, `((()))` $\in L_2$, `(()())` $\in L_2$, and `(((())))(())` $\in L_2$, but `((((()))))` $\notin L_2$ because although it's a string of balanced parentheses, the nesting goes five levels deep.

ii. Design a DFA for L_2, showing that L_2 is regular. A consequence of *this* result is that while you can't parse all programs or HTML with regular expressions, you can parse programs with low nesting depth or HTML documents without deeply-nested tags using regexes. ***Please submit this DFA using the DFA editor on the course website*** and tell us on GradeScope who submitted it.

iii. Look back at your proof from part (i) of this problem. Imagine that you were to take that exact proof and blindly replace every instance of "L_1" with "L_2." This would give you a (incorrect) proof that L_2 is nonregular (which we know has to be wrong because L_2 is indeed regular.) Where would the error be in that proof? Be as specific as possible.

iv. Intuitively, regular languages correspond to problems that can be solved using only finite memory. Using this intuition and without making reference to DFAs, NFAs, or the Myhill-Nerode theorem, explain why L_1 is nonregular while L_2 is regular.

Problem Six: Tautonyms

A ***tautonym*** is a word that consists of the same string repeated twice. For example, "caracara" and "dikdik" are all tautoynms (the first is a bird, the second the cutest animal you'll ever see), as is "hotshots" (people who aren't very fun to be around). Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and consider the following language:

$$L = \{ \ ww \mid w \in \Sigma^* \ \}$$

This is the language of all tautonyms over Σ. Below is an ***incorrect*** proof that L is not regular:

Proof: Let $S = \{ \ \mathtt{a}^n \mid n \in \mathbb{N} \ \}$. This set is infinite because it contains one string for each natural number. We claim that any two strings in S are distinguishable relative to L. To see this, consider any two distinct strings $\mathtt{a}^m$ and $\mathtt{a}^n$ in the set S, where $m \neq n$. Then $\mathtt{a}^m\mathtt{a}^m \in L$ but $\mathtt{a}^n\mathtt{a}^m \notin L$, so $\mathtt{a}^m \not\equiv_L \mathtt{a}^n$. This means that S is an infinite set of strings that are pairwise distinguishable relative to L. Therefore, by the Myhill-Nerode theorem, L is not regular. ■

i. What's wrong with this proof? Be as specific as possible.

ii. Although the above proof is incorrect, the language L isn't regular. Prove this. Please make sure that your proof doesn't make the same error that you identified above!

Problem Seven: State Lower Bounds

The Myhill-Nerode theorem we proved in lecture is actually a special case of a more general theorem about regular languages that can be used to prove lower bounds on the number of states necessary to construct a DFA for a given language.

i. Let L be a language over Σ. Suppose there's a *finite* set S such that any two distinct strings $x, y \in S$ are distinguishable relative to L (that is, $x \not\equiv_L y$). Prove that any DFA for L must have at least $|S|$ states. (You sometimes hear this referred to as *lower-bounding* the size of any DFA for L.)

According to old-school Twitter rules, all tweets need to be 140 characters or less. Let Σ be the alphabet of characters that can legally appear in a tweet and consider the following language:

$$TWEETS = \{ w \in \Sigma^* \mid |w| \le 140 \}.$$

This is the language of all legal tweets. The good news is that this language is regular. The bad news is that any DFA for it has to be pretty large.

ii. Using your result from part (i), prove that any DFA for *TWEETS* must have at least 142 states.

iii. Refer back to the formal, 5-tuple definition of a DFA given in Problem Set Six. Define an 142-state DFA for *TWEETS* using the formal 5-tuple definition. Briefly explain how your DFA works. No formal proof is necessary.

Your results from parts (ii) and (iii) collectively establish that the smallest possible DFA for *TWEETS* has exactly 142 states. This approach to finding the smallest object of some type – using some theorem to prove a lower bound ("we need at least this many states") combined with a specific object of the given type ("we certainly can't do worse than this") is a common strategy in discrete mathematics, algorithm design, and computational complexity theory. If you take classes like CS161, CS254, etc., you'll likely see similar sorts of approaches!

Problem Eight: Closure Properties Revisited

When building up the regular expressions, we explored several closure properties of the regular languages. This problem explores some of their nuances.

The regular languages are closed under complementation: If L is regular, so is $\overline{L}$.

i. Prove or disprove: the *nonregular* languages are closed under complementation.

The regular languages are closed under union: If L_1 and L_2 are regular, so is $L_1 \cup L_2$.

ii. Prove or disprove: the *nonregular* languages are closed under union.

We know that the union of any two regular languages is regular. Using induction, we can show that the union of any finite number of regular languages is also regular. As a result, we say that the regular languages are closed under *finite union*.

An *infinite union* is the union of infinitely many sets. For example, the rational numbers can be expressed as the infinite union $\{ {}^{x}/_{1} \mid x \in \mathbb{Z} \} \cup \{ {}^{x}/_{2} \mid x \in \mathbb{Z} \} \cup \{ {}^{x}/_{3} \mid x \in \mathbb{Z} \} \cup \ldots$ out to infinity.

iii. Prove or disprove: the regular languages are closed under infinite union.

Optional Fun Problem: Generalized Fooling Sets (1 Point Extra Credit)

In Problem Seven, you saw how to use distinguishability to lower-bound the size of DFAs for a particular language. Unfortunately, distinguishability is not a powerful enough technique to lower-bound the sizes of NFAs. In fact, it's in general quite hard to bound NFA sizes; there's a $1,000,000 prize for anyone who finds a polynomial-time algorithm that, given an arbitrary NFA, converts it to the smallest possible equivalent NFA!

Although it's generally difficult to lower-bound the sizes of NFAs, there are some techniques we can use to find lower bounds on the sizes of NFAs. Let L be a language over Σ. A ***generalized fooling set*** for L is a set $\mathscr{F} \subseteq \Sigma^* \times \Sigma^*$ is a set with the following properties:

- For any $(x, y) \in \mathscr{F}$, we have $xy \in L$.
- For any distinct pairs (x_1, y_1), $(x_2, y_2) \in \mathscr{F}$, we have $x_1y_2 \notin L$ or $x_2y_1 \notin L$ (this is an inclusive OR.)

Prove that if L is a language and there is a generalized fooling set $\mathscr{F}$ for L that contains n pairs of strings, then any NFA for L must have at least n states.